An Alternate Model for Uplink Interference in CDMA Systems with Power Control

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***Abstract*—Accurately modeling the statistics of the uplink interference power is an important problem in the design and analysis of second and third generation code division multiple access cellular systems. In the uplink, the total interference varies not only because of wireless channel effects such as shadowing and path loss, but also because of the randomness in the number of interfering mobiles and their locations. Conventionally, it has been argued this can be well modeled as a Gaussian distribution, especially in the presence of power control. In this paper, we show that when all the sources of randomness are considered together, the uplink inter-cell interference power is better modeled by a lognormal distribution. Using the well established theory of Poisson point processes to model spatial mobile distributions, we extend the moment matching or Fenton-Wilkinson method to determine the parameters of the approximating lognormal. Our results show that our lognormal approximation is several orders of magnitude more accurate than the conventional Gaussian approximation. These results have applications in cell planning and layout and, in general, in cellular system design and analysis.**

I. Introduction

Interference plays a crucial role in code division multiple access (CDMA) based cellular communication systems. This is because of the use of pseudo-random spreading codes for transmitting data. While the spreading codes diminish the interference received from other transmissions, they do not completely annul it. The uplink interference consists from intra-cell interference and inter-cell interference. The uplink inter-cell interference signal received by a base station is the sum of interference signals from the many interfering users served by other base stations. The net power is a random variable (RV) because the interfering signals undergo shadowing and fading in their respective wireless channels. Therefore, cellular system design and analysis requires an accurate statistical characterization of the interference power.

In the literature, the uplink inter-cell interference power has been often modeled by a Gaussian RV, with the central limit theorem being cited as a justification for this [1]. Furthermore, it has been argued that the Gaussian approximation becomes more accurate in the presence of power control and cell selection [2]. In this paper, we show that the Gaussian approximation does not accurately model the uplink inter-cell interference even in the presence of power control. Instead, we show that the lognormal distribution is significantly more accurate.

Intuitively, this can be understood as follows. The interfering signal from each user undergoes lognormally distributed shadowing [3]. When the number of interfering signals and their shadowing parameters is deterministically known, it is well understood that the sum of lognormal RVs is well approximated by a lognormal RV [4], [5].[1] In other words, while the distribution of the sum does eventually become a Gaussian RV, the rate of convergence of the sum distribution to the Gaussian is slow. Consequently, several methods, such as the Fenton-Wilkinson moment matching method [7], the Schwartz-Yeh log-moment matching method [8], the Schleher cumulant matching method [9], and more recently, the Beaulieu-Xie characteristic function inversion method [10] and the Mehta et. al. moment generating function matching method [6] have been proposed for determining the parameters of the approximating lognormal.

However, modeling the uplink CDMA inter-cell interference poses a new twist to this problem since the number of interfering mobiles itself is random. Furthermore, additional randomness is introduced because the transmitting mobiles can be located anywhere within a cell. The use of power control adds an additional dimension to this problem since it affects the power transmitted by the mobiles.

In this paper, we use the elegant Poisson point process theory to model the spatial randomness of the interfering mobiles. The theory provides a tractable and reasonable model for the spatial randomness observed in a CDMA uplink, and has been used effectively in several wireless system design problems [1], [11]. Based on this, we extend the moment matching method, initially proposed by Fenton-Wilkinson for the case when the number of interferers is deterministic, to analytically determine the parameters of the approximating lognormal. As we shall see, despite the additional sources of randomness, the simplicity and accuracy of the F-W method carry over to our problem a large extent. We show that closed-form expressions, in terms of a single integral of a simple function, can be written for the parameters of the

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[1] An approximation is necessary in the first place because a closed-form expression for the probability distribution of a sum of lognormal RVs is unknown, except for certain special cases [6].

approximating lognormal.[2]

Our results thus provide a more accurate snapshot model for the interference. Consequently, they have applications in cell planning and layout, and, in general, cellular system design and analysis. It must be noted that while this model is useful, it is not entirely sufficient. For example, to analyze call session or data session specific behavior, a more detailed time trace model or a time correlation model (in addition to the snapshot) would also be needed. However, this is well beyond the scope of this paper.

The paper is organized as follows. The system model is developed in Sec. II. The alternate lognormal model for interference is developed in Sec. III. Simulation results are presented in Sec. IV, and are followed by our conclusions in Sec. V.

II. Uplink CDMA System Model Basics

Figure 1 shows the hexagonal cellular layout consisting of a cell 0, which we henceforth refer to as the *reference cell*, and an adjacent interfering cell k. The cell k is served by base station (BS) k, which is located at the cell's center. If first-tier interferers are considered, k takes a value from 1 to 6, and if second-tier interferers are also considered, k additionally takes values from 7 to 18. Let D_k denote the distance between BS k and the reference BS 0. We focus on the fading-averaged interference case. While including the short-term fading in the model is desirable, it is beyond the scope of this paper.

Consider a mobile i inside cell k. Let $\mathbf{x}_i(k)$ denote the position vector of mobile i with respect to BS k. Let $\|\mathbf{x}_i(k)\|$ denote the corresponding distance of the mobile from BS k.

When the mobile i transmits a signal with power P_i, the short-term fading-averaged receive signal power at BS k is given by

$$R_i(k) = P_i \left(\frac{d_0}{\|\mathbf{x}_i(k)\|}\right)^{\epsilon} s_i^{(k)}, \quad (1)$$

where d_0 is a reference distance and ϵ is the path loss coefficient, which typically takes values between 2 and 4 [3]. The variable $s_i^{(k)}$ denotes the shadowing of the uplink channel from mobile i to BS k. As mentioned, shadowing is well characterized by a lognormal random variable (RV), and can be written as

$$s_i^{(k)} = 10^{0.1 y_i(k)} = e^{\beta y_i(k)}, \quad (2)$$

where $y_i(k)$ is a Gaussian RV with zero mean and variance $\sigma_i^2(k)$ and $\beta = 0.1 \log_e(10)$. Following terminology used in the literature, we shall refer to $\sigma_i^2(k)$ as the dB variance of the lognormal RV $s_i^{(k)}$. Typically, $\sigma_i(k)$ takes values between 4 and 12. We assume $y_i(k)$ to be independent and identically distributed for different values of i and k.

For a mobile i, let its serving cell, which controls and decodes the mobile's transmissions, be denoted by $C(i)$. In this paper, we assume that the serving cell is the geographically nearest cell. (Our analysis can also be generalized to handle the case of cell selection. But, this is beyond the scope of this paper.) In the presence of power control, each transmitting mobile regulates its transmit power (using feedback from the serving BS) so that the receive power equals a preset threshold, γ. Therefore, $R_i(C(i)) = \gamma$. From (1), we get

$$P_i = \frac{\gamma}{s_i^{(C(i))}} \left(\frac{d_0}{\|\mathbf{x}_i(C(i))\|}\right)^{-\epsilon}. \quad (3)$$

Hence, if $C(i) \neq 0$, the interference power received by the reference BS 0 from mobile i equals

$$R_i(0) = \gamma \frac{s_i^{(0)}}{s_i^{(C(i))}} \left(\frac{\|\mathbf{x}_i(C(i))\|}{\|\mathbf{x}_i(0)\|}\right)^{\epsilon}, \quad (4)$$

$$= \gamma e^{\beta(y_i(0) - y_i(C(i)))} \left(\frac{\|\mathbf{x}_i(C(i))\|}{\|\mathbf{x}_i(0)\|}\right)^{\epsilon}. \quad (5)$$

The second step follows from (2). Note that $y_i(0) - y_i(C(i))$ is also a Gaussian RV with zero mean and variance $\sigma_i^2(0) + \sigma_i^2(C(i))$. Henceforth, for analytical simplicity, we shall assume that $\sigma_i(k)$ is the same for all users and cells, i.e., $\sigma_i(k) = \sigma$. The method easily generalizes to the unequal $\sigma_i(k)$ case, albeit with the help of some extra book-keeping notation.

Let $I_k\left(N_k; \{\mathbf{x}_i(k)\}_{i=1}^{N_k}\right)$ denote the total inter-cell interference power at BS 0 from users served by BS k, given that the number of interfering users is N_k and their locations are $\mathbf{x}_1(k), \ldots, \mathbf{x}_{N_k}(k)$. Then

$$I_k\left(N_k; \{\mathbf{x}_i(k)\}_{i=1}^{N_k}\right) = \gamma \sum_{i=1}^{N_k} e^{\beta(y_i(0) - y_i(C(i)))} \left(\frac{\|\mathbf{x}_i(C(i))\|}{\|\mathbf{x}_i(0)\|}\right)^{\epsilon}. \quad (6)$$

A. Poisson Point Process Model for Users

The Poisson point process model provides an analytically tractable model for the random user locations in a cell area. Briefly, a homogeneous point process is characterized by an intensity parameter λ. The probability that N_k users occur within a cell of area A follows the Poisson distribution with mean λA, and equals

$$\Pr(N_k) = \frac{(\lambda A)^{N_k}}{N_k!} \exp(-\lambda A). \quad (7)$$

Furthermore, conditioned on N_k, the geographical locations of the N_k mobiles are uniformly distributed over the cell area. While we limit our attention in this paper to homogeneous Poisson point processes, the analysis can be extended to handle non-homogeneous processes as well.

This implies that with power control, the intra-cell interference power received by any cell of area A from the users it serves is also a Poisson random variable with mean $\gamma\lambda A$ [1]. Therefore, only the probability distribution of the inter-cell interference remains to be characterized, as done in the next section.

[2]Modeling the intra-cell interference is relatively easy given a Poisson point process model [1], and is therefore not considered in this paper. This is discussed briefly in Sec. II.

III. Alternate Interference Model

The problem at hand is to characterize the distribution of the total interference from cell k, denoted by I_k, when unconditioned on the random number of users and their locations in the cell. As mentioned, even for a fixed number of interferers, an expression for the probability distribution of $I_k(.;.)$ is not available in closed-form.

We now develop the moment matching method to approximate the distribution of the total inter-cell interference power received by reference BS 0 from BS k by a lognormal RV, I_{eq}. Note that the same methodology directly applies when the distribution of the total interference power received by reference BS 0 from all its neighboring interfering cells needs to be characterized. For simplicity, we henceforth drop the interfering cell index k from the notation unless required otherwise.

The moment matching method determines the parameters of I_{eq} by matching its mean and variance with those of the total interference, I, that is actually received by the reference BS 0. The approximating lognormal RV can be written as $I_{\text{eq}} = \exp(\beta y_{\text{eq}})$, where y_{eq} is a Gaussian RV with mean μ_{eq} and variance σ^2_{eq}. Then, it can be shown that,

$$\mathbf{E}\left[I_{\text{eq}}\right] = \exp(\beta\mu_{\text{eq}} + \beta^2\sigma^2_{\text{eq}}/2), \quad (8)$$

$$\mathbf{E}\left[I^2_{\text{eq}}\right] = \exp(2\beta\mu_{\text{eq}} + 2\beta^2\sigma^2_{\text{eq}}), \quad (9)$$

where $\mathbf{E}[.]$ denotes the expectation operator. Equating these two expressions with the first and second moments of I yields

$$\mu_{\text{eq}} = \log_e\left(\frac{\mathbf{E}^2[I]}{\sqrt{\mathbf{E}[I^2]}}\right), \quad (10)$$

$$\sigma_{\text{eq}} = \sqrt{\log_e\left(\frac{\mathbf{E}[I^2]}{\mathbf{E}^2[I]}\right)}. \quad (11)$$

We now proceed to derive expressions for the first and second moments of the total interference I, when averaged over the shadowing as well as the spatial Poisson point process model.

A. Evaluating First and Second Moments of I

In this section, we show the following two main results:

$$\mathbf{E}[I] \approx 2\gamma N_{\text{ave}}\left(\frac{D_k}{R}\right)^2 e^{\beta^2\sigma^2} \times \frac{1}{W}\sum_{w=1}^{W}\int_{\frac{D_k}{R}}^{\infty}\frac{1}{u^3}\left(1+u^2-2ua_w\right)^{-\epsilon/2} du, \quad (12)$$

and

$$\mathbf{E}\left[I^2\right] \approx 2\gamma N_{\text{ave}}\left(\frac{D_k}{R}\right)^2 e^{4\beta^2\sigma^2} \times \frac{1}{W}\sum_{w=1}^{W}\int_{\frac{D_k}{R}}^{\infty}\frac{1}{u^3}\left(1+u^2-2ua_w\right)^{-\epsilon} du, \quad (13)$$

where $N_{\text{ave}} = \lambda A$ is the average number of users in the interfering cell of area A, and a_w, $1 \le w \le W$, are the abscissa of Gauss-Chebyshev quadrature [12].

From (6) and the assumption that the shadowing is independent of user location, we have

$$\mathbf{E}\left[I(N;\{\mathbf{x}_i(k)\}_{i=1}^N)|N\right] = \gamma\sum_{i=1}^{N}\mathbf{E}\left[e^{\beta(y_i(0)-y_i(C(i)))}\right]\mathbf{E}\left[\left(\frac{\|\mathbf{x}_i(C(i))\|}{\|\mathbf{x}_i(0)\|}\right)^{\epsilon}\right], \quad (14)$$

where $\mathbf{E}[.|N]$ denotes conditional expectation given the number of interfering users N. For a homogeneous Poisson point process, the user locations are uniformly distributed over the cell area given N. Therefore, the interference averaged over the user locations, conditioned on N, is given by

$$\mathbf{E}[I(N)|N] = N\gamma e^{\beta^2\sigma^2}\mathbf{E}\left[\left(\frac{\|\mathbf{x}_i(C(i))\|}{\|\mathbf{x}_i(0)\|}\right)^{\epsilon}\right], \quad (15)$$

where i is an arbitrary user *served* by BS k.

For analytical tractability, we approximate a hexagon with a circle of radius R. Since a user's location is uniformly distributed over the cell area, we have

$$p(r,\phi_i) = \frac{rdr}{\pi R^2}d\phi_i, \quad (16)$$

where $r = \|\mathbf{x}_i(k)\|$ and ϕ_i is the azimuth of the user's location as shown in Fig. 1. Therefore, $\|\mathbf{x}_i(0)\| = \sqrt{r^2 + D_k^2 - 2rD_k\cos(\phi_i)}$. Hence,

$$\begin{aligned}\mathbf{E}&\left[\left(\frac{\|\mathbf{x}_i(C(i))\|}{\|\mathbf{x}_i(0)\|}\right)^{\epsilon}\right] \\ &= \frac{1}{\pi R^2}\int_0^{2\pi}\int_0^{R} r\left(1+\frac{D_k^2}{r^2}-2\frac{D_k}{r}\cos(\phi_i)\right)^{-\epsilon/2} dr\, d\phi_i, \\ &= \frac{2D_k^2}{\pi R^2}\int_{D_k/R}^{\infty}\int_{-1}^{1}\frac{1}{u^3\sqrt{1-z^2}}(1+u^2-2uz)^{-\epsilon/2}\, dz\, du.\end{aligned} \quad (17)$$

where $u = D_k/r$ and $z = \cos(\phi_i)$. Using Gauss-Chebyshev quadrature up to W terms [13], we get

$$\mathbf{E}\left[\left(\frac{\|\mathbf{x}_i(C(i))\|}{\|\mathbf{x}_i(0)\|}\right)^{\epsilon}\right] \approx \frac{2D_k^2}{WR^2}\sum_{w=1}^{W}\int_{D_k/R}^{\infty}\frac{1}{u^3}(1+u^2-2ua_w)^{-\epsilon/2}\, du, \quad (18)$$

where a_w, $1 \le w \le W$, are the first W abscissa of Gauss-Chebyshev quadrature. The error in the approximation decreases to 0 as W increases. ($W = 12$ turned out to be more than sufficient in our problem.) Substituting the above result in (15) and unconditioning over N results in the desired expression in (12).

The derivation for $\mathbf{E}\left[I^2\right]$ follows along similar lines, and is not repeated here.

IV. Simulations

We now plot the cumulative distribution function (CDF) and complementary CDF (CCDF) of the measured uplink interference from a first-tier interfering cell. For this purpose, Monte Carlo simulations were used to generate 7×10^6 sample points. Also plotted are the CDF and CCDF of the analytically approximated interference assuming a Gaussian distribution and assuming a lognormal distribution, with parameters derived using the results of the previous section. The mean and variance of the Gaussian distribution are obtained by equating them with the mean and variance of the total interference, along lines similar to that in Sec. III-A. Plotting and comparing the CDF and CCDF is instructive because small values of the CDF reveal the accuracy in tracking the head portion (small inter-cell interference values) of the probability distribution, while small values of the CCDF reveal the accuracy in tracking the tail portion (large inter-cell interference values) of the probability distribution.

The following system parameters were used in the simulations: path loss exponent, $\epsilon = 4$, power control threshold, $\gamma = 8$ dB, and lognormal dB standard deviation, $\sigma = 6$. The cell radius was taken to be $R = 400$ m and the first-tier inter-BS distance was $D_k = 800$ m.

Figure 2 plots the CDF when the average number of users per cell is $N_{\text{ave}} = 10$ (which corresponds to $\lambda = N_{\text{ave}}/(\pi R^2)$). Figure 3 plots the same when the average number of users per cell is larger and equals 30. It can be seen from both the plots that the simulated CDF of the uplink interference decays considerably faster than a Gaussian as the interference (x-axis) tends to 0. In both figures, the Gaussian approximation is inaccurate by two orders of magnitude. On the other hand, the proposed lognormal approximation method is able to track the observed CDF much better.

The difference in the behavior of the Gaussian and lognormal CDFs for small values of the interference can be understood as follows. The probability that a Gaussian RV, with mean μ_G and variance σ_G^2, is less than x is given by $1 - Q\left(\frac{x-\mu_G}{\sigma_G}\right)$. As $x \to 0^+$, this saturates at $Q\left(\frac{\mu_G}{\sigma_G}\right)$.[3] On the other hand, the probability that a lognormal RV, with dB mean μ_L and dB variance σ_L^2, is less than x is given by $1 - Q\left(\frac{\log_e(x)-\mu_L}{\sigma_L}\right)$. For small x, this equals $Q\left(\frac{|\log_e(x)|}{\sigma_L}\right)$, which tends to 0 since $|\log_e(x)| \to \infty$ as $x \to 0^+$.

The CDF figures above show that despite having an accuracy considerably better than the Gaussian approximation, the moment matching method based lognormal approximation is clearly not perfect. Furthermore, the accuracy decreases as the average number of users per cell increases. This result is in line with the observations made in the literature for the case when the number of lognormal summands (and their parameters) is deterministic [4], [6], [10]. It is known that the moment matching method emphasizes a more accurate fit for the tail portion of the probability distribution than its head portion.

Figures 4 and 5 plot the corresponding CCDF curves when the average number of users per cell is 10 and 30, respectively. The lognormal approximation obtained using the moment matching method tracks the actual CCDF very well. Once again we observe that the lognormal approximation is significantly better than the Gaussian approximation. This

[3]It can be shown that the mean of the Gaussian RV approximating the interference is positive.

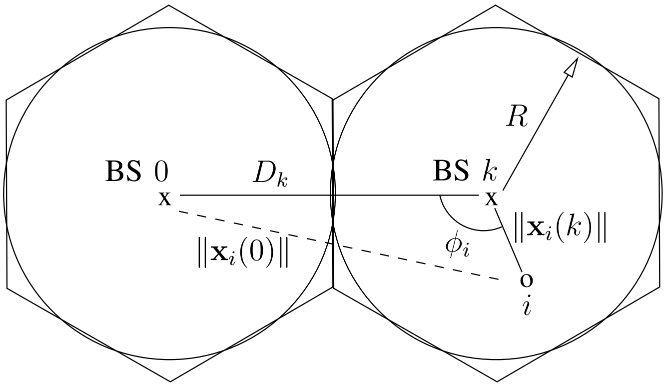

Fig. 1. Hexagonal cellular layout showing the reference cell 0, a first-tier interfering cell k, and the relative position of an interfering user i served by cell k.

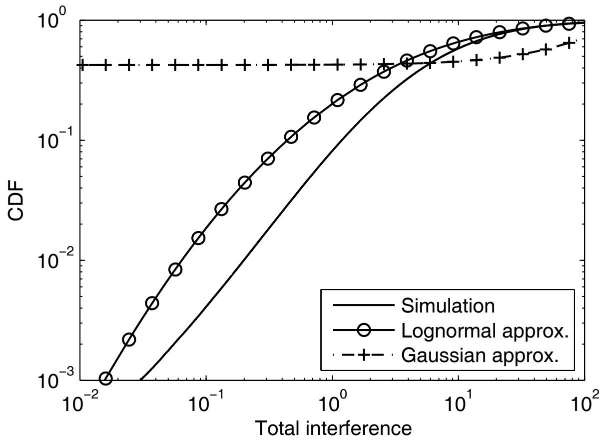

Fig. 2. Comparison of accuracy of CDFs of uplink interference power obtained from the proposed lognormal approximation method and the Gaussian approximation when the average number of users per cell is 10.

is also in line with the observations made in the literature for the case when the number of lognormal summands is deterministic [4], [6], [10].

V. Conclusions

In this paper, we developed an alternate characterization of the statistical distribution of the inter-cell interference seen in the uplink of CDMA systems that use power control. In the uplink, randomness is introduced in the interference power not only because of lognormal shadowing but also because of random number of users in the cells and their random spatial locations. The moment matching lognormal approximation method turned out to be several orders of magnitude more accurate than the conventional Gaussian approximation even when the number of number of interfering users was relatively large. Both the head portion and tail portion of the inter-cell interference probability distribution were better approximated by a lognormal RV than a Gaussian RV.

The proposed method is easy to implement as the parameters of the approximating lognormal can be written in closed-

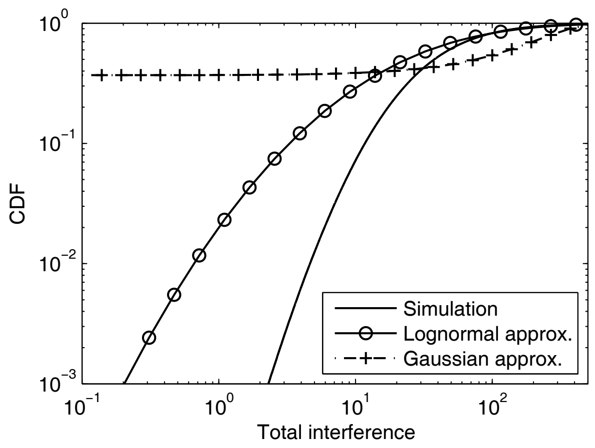

Fig. 3. Comparison of accuracy of CDFs of uplink interference power obtained from the proposed lognormal approximation method and the Gaussian approximation when the average number of users per cell is 30.

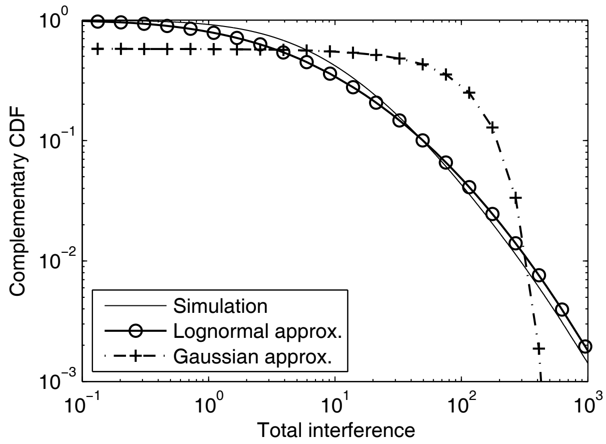

Fig. 4. Comparison of accuracy of CCDFs of uplink interference power obtained from the proposed lognormal approximation method and the Gaussian approximation when the average number of users per cell is 10.

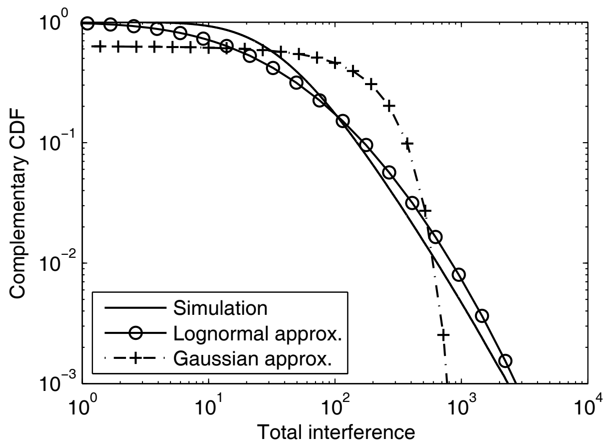

Fig. 5. Comparison of accuracy of CCDFs of uplink interference power obtained from the proposed lognormal approximation method and the Gaussian approximation when the average number of users per cell is 30.

form in terms of the underlying wireless channel parameters and the parameters of the Poisson point process driving the spatial distribution of users. Our results also showed that several insights for the case where the number of lognormal summands was fixed (and so were their parameters) carried over to the uplink scenario as well. While the results obtained are considerably more accurate than those typically used in the literature, scope for further improvement still remains. Future work involves improving the lognormal approximation method to better match CDF and CCDF.

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